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Determination of shear viscosity and shear rate from pressure drop and flow rate relationship in a rectangular channel

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Abstract

In this study, we present a unique approach to calculate the shear viscosity and shear rate with the pressure drop and flow rate data from a channel having a rectangular cross-section with a height-to-width ratio (H/W) of close to one. The derived equation was verified with experimental data from rectangular dies whose height-to-width ratio (H/W) ranges from 0.1 to 1. It was confirmed that the proposed approach is reliable for the calculation of the shear viscosity and shear rate from the flow data in a rectangular channel.

Keywords: Rectangular channel; Pressure flow; Non-Newtonian fluids

1. Introduction

The capillary rheometer and the slit rheometer are the most commonly employed apparatuses for the measurement of the viscosity of a polymer melt, since they provide simple measurement and a high shear rate range [1]. Other rheometers hardly provide a shear rate beyond 500 s^{-1}. This shear rate range is common in polymer processing, including injection molding, extrusion molding, transfer molding, etc.

In the tests, polymer melts are forced to flow by a piston under a given pressure (or by screw rotation in an extruder) through a capillary or a slit die. Raw data in a capillary and a slit viscometer are the volumetric flow rate Q and the required pressure drop across the die length (a pressure gradient) $\Delta P/L$ at a given temperature. The flow rate and the pressure gradient are easily converted to shear rate and shear stress by a well-known procedure, as will be shown in Section 2. The difference between the capillary and the slit viscometer lies in the shape of the cross-section of the die: the capillary die has a circular cross-section whereas the slit die has a wide rectangular cross-section.

In order to obtain accurate experimental results, the height-to-width ratio (H/W) of the slit die should be nearly zero ($W \gg H$), because the existing theoretical equation converting Q and $\Delta P/L$ to $\dot{\gamma}_w$ and τ_w is based on the assumption that the velocity does not depend on the lateral position. Thus, the edges make a negligible contribution to the pressure drop. However, under certain conditions, this requirement is not met. For example, many researchers use rectangular ducts, where the height (H) is comparable to the width (W), to investigate the flow properties in micro-channels whose dimensions range from several microns to several hundreds of microns [2–5] and the flow properties of suspension [6]. In these cases, the exact shear rate cannot be obtained from the volumetric flow rate because there is no explicit relation between Q and $\dot{\gamma}_w$ in this geometry. To the best of our knowledge, there is no method that enables conversion of the ΔP and Q relation to τ_w and $\dot{\gamma}_w$ in a rectangular duct whose height is comparable to the width.

In this paper, we propose an explicit relationship to obtain the exact shear rate and shear stress relationship from the flow data (i.e., ΔP and Q) in a rectangular duct having H that is comparable to W.

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2. Theoretical analysis

2.1. Capillary die

Capillary viscosity measurements are based on the relation between the pressure drop across the capillary and the flow rate. The wall shear stress τ_w can be directly obtained from the pressure drop and the geometric constants (capillary radius R and die length L). At a steady laminar flow, the sum of all the forces acting on the cylindrical element, as shown in Fig. 1, is zero, i.e.,

$$\sum F = F_1 + F_2 + F_3 \tag{1}$$

$$F_1 \text{ (a hydrostatic force)} = P(\pi R^2) \tag{2}$$

$$F_2 \text{ (a hydrostatic force)} = -\left(P + \left(\frac{\partial P}{\partial z}\right)\mathrm{d}z\right)\pi R^2 \tag{3}$$

where $(\partial P/\partial z)$ is an expression of the change in the pressure P with position in the z direction and

$$\begin{aligned} &F_3 \text{ (the drag on the surface of die wall)} \\ &\quad = \text{surface area} \times \text{drag/unit surface} \\ &\quad = \text{surface area} \times \text{shear stress at the wall} = 2\pi R^2 \mathrm{d}z\tau_w \end{aligned} \tag{4}$$

Substituting Eqs. (2)–(4) into Eq. (1) and rearranging, we obtain

$$\tau_w = \frac{R}{2}\left(\frac{\partial P}{\partial z}\right) \tag{5}$$

Common experience leads to the expectation that the pressure gradient $(\partial P/\partial z)$ is independent of the position, z. Hence, if the total pressure drop between the ends of the capillary of length L is ΔP, the following equation can be obtained:

$$\tau_w = \frac{R}{2}\left(\frac{\Delta P}{L}\right) \tag{6}$$

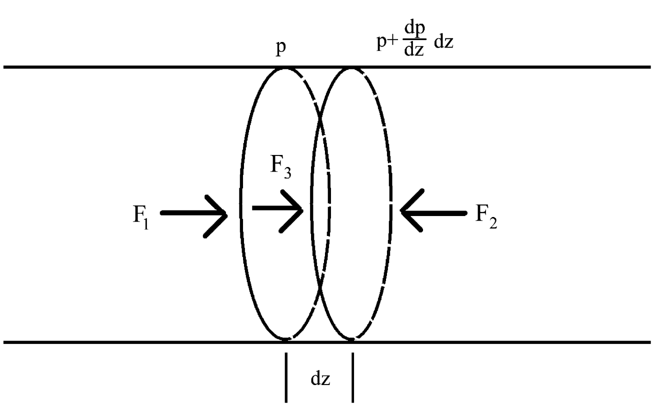

Fig. 1. Balance of forces on fluid element moving along with flow in a pipe or tube.

For Newtonian fluids, the wall shear rate is readily obtained through the use of the Hagen–Poiseuille relation [1]. It takes the following form after some rearrangement.

$$\frac{R\Delta P}{2L} = \mu\left(\frac{4Q}{\pi R^3}\right) \tag{7}$$

where μ and Q are the viscosity of Newtonian fluids and the flow rate through the capillary tube, respectively. Then, from Eq. (6) and the constitutive relation, shear stress = viscosity × shear rate, we obtain

$$\dot{\gamma}_a = \left(\frac{4Q}{\pi R^3}\right) \left(\text{wall shear rate of Newtonian fluid or apparent shear rate}\right) \tag{8}$$

The derivation of the wall shear rate is slightly more complex in the case of non-Newtonian fluids. Specifically, it cannot be obtained from a single data point. For non-Newtonian fluids, the wall shear rate can be obtained by the Rabinowitsch relation [1]. The detailed procedure for the derivation of the Rabinowitsch relation is not presented here; only final forms are shown.

$$\dot{\gamma}_w = \frac{3n+1}{4n}\left(\frac{4Q}{\pi R^3}\right) = \frac{3n+1}{4n}\dot{\gamma}_a$$
$$\text{where } n = \frac{\mathrm{d}(\log \Delta P)}{\mathrm{d}(\log Q)} = \frac{\mathrm{d}(\ln \Delta P)}{\mathrm{d}(\ln Q)} = \frac{\mathrm{d}(\log \tau_w)}{\mathrm{d}(\log \dot{\gamma}_a)} = \frac{\mathrm{d}(\ln \tau_w)}{\mathrm{d}(\ln \dot{\gamma}_a)} \tag{9}$$

The wall shear rate can be obtained indirectly from a log–log plot of the wall shear stress versus the apparent shear rate. The curve is frequently close to a straight line, and the slope n is a constant equal to the power-law index.

Eq. (9) can be alternatively derived by the following approach. For a power-law fluid, we can easily derive the following relation between ΔP and Q by a flow analysis in a capillary die [7].

$$\frac{R\Delta P}{2L} = K\left(\frac{3n+1}{4n}\frac{4Q}{\pi R^3}\right)^n \tag{10}$$

where K is the consistency index of a power-law fluid.

By combining the definition of the wall shear stress, given by Eq. (6), and the apparent shear rate, given by Eq. (8), Eq. (10) becomes

$$\tau_w = K\left(\frac{3n+1}{4n}\dot{\gamma}_a\right)^n \tag{11}$$

Then, with the constitutive relation of the power-law fluid, $\tau_w = K\dot{\gamma}_w^n$, we can obtain the Rabinowitsch relation, Eq. (9) from Eq. (11).

2.2. Slit die

A slit die is a rectangular duct whose width W is infinitely larger than the height H ($W \gg H$ or $H/W = 0$). An analogous approach with Eqs. (1)–(4) leads to the relation between ΔP and τ_w in a slit die of length L.

$$\Delta P \times \text{cross sectional area of the slit} = \tau_w \times \text{area of the die wall}$$

$$\Delta P \times (WH) = \tau_w \times (2WL + 2HL) \approx \tau_w \times (2WL) \tag{12}$$

This, when rearranged, simplifies to

$$\tau_w = \frac{H}{2}\left(\frac{\Delta P}{L}\right) \tag{13}$$

For Newtonian fluids, the relation between ΔP and Q is [1,7]

$$\frac{H\Delta P}{2L} = \mu\left(\frac{6Q}{WH^2}\right) \tag{14}$$

Therefore, the apparent shear rate for the slit die is

$$\dot{\gamma}_a = \left(\frac{6Q}{WH^2}\right) \tag{15}$$

For power-law fluids, the relation between ΔP and Q is [1,7]

$$\frac{H\Delta P}{2L} = K\left[\left(\frac{2n+1}{3n}\right)\left(\frac{6Q}{WH^2}\right)\right]^n \tag{16}$$

Hence, the wall shear rate of non-Newtonian fluids in a slit die is

$$\dot{\gamma}_w = \frac{2n+1}{3n}\left(\frac{6Q}{WH^2}\right) = \frac{2n+1}{3n}\dot{\gamma}_a \tag{17}$$

2.3. Rectangular die

From Eq. (12), we can obtain the relation between τ_w and ΔP in a rectangular die where H is comparable to W.

$$\tau_w = \left(\frac{\Delta P}{2L}\right)\left(\frac{WH}{H+W}\right) = \left(\frac{\Delta PH}{2L}\right)\left(\frac{1}{H/W+1}\right) \tag{18}$$

Explicitly speaking, the wall shear stress in Eq. (18) is an average with respect to the perimeter of the rectangular duct, because τ_w varies with the position of the die wall.

In order to derive the explicit relation between $\dot{\gamma}_w$ and $\dot{\gamma}_a$, a flow analysis of the rectangular ducts should be conducted for both Newtonian fluids and non-Newtonian fluids. For Newtonian fluids, an analytical solution can be derived for a steady laminar flow in a rectangular duct [8]. After some rearrangement, the result takes the following form.

$$\left(\frac{\Delta PH}{2L}\right)\left(\frac{1}{H/W+1}\right) = \mu\left(\frac{6Q}{WH^2}\right)\left(1+\frac{H}{W}\right)f^*\left(\frac{H}{W}\right) \tag{19}$$

where the function $f^*(x)$ is given as follows.

$$f^*(x) = \left[\left(1+\frac{1}{x}\right)^2\left(1-\frac{192}{\pi^5 x}\sum_{i=1,3,5}^{\infty}\frac{\tanh(\frac{\pi}{2}ix)}{i^5}\right)\right]^{-1} \tag{20}$$

Therefore, the apparent shear rate of a rectangular die is

$$\dot{\gamma}_a = \left(\frac{6Q}{WH^2}\right)\left(1+\frac{H}{W}\right)f^*\left(\frac{H}{W}\right) \tag{21}$$

The function $f^*(x)$ is calculated and shown in Table 1. When the aspect ratio H/W approaches zero, Eq. (21) becomes Eq. (15) for a slit die.

For power-law fluids, an analytical solution cannot be obtained in the rectangular geometry. Instead, the relation between ΔP and Q is obtained by a numerical calculation. Several authors have performed the numerical calculations for power-law fluids in a rectangular channel [9,10]. Hartnett and Kostic [11] obtained the ΔP–Q relation over a full range of aspect ratio (H/W) and power-law index (n). They reported the following useful expression from their numerical calculation results.

$$\left(\frac{\Delta P}{2L}\right)\left(\frac{WH}{H+W}\right) = K\left[\left(\frac{6Q}{WH^2}\right)\left(1+\frac{H}{W}\right)\left(\frac{2}{3}\right)\left(b^*+\frac{a}{n}\right)\right]^n \tag{22}$$

where a^* and b^* are functions that depend only on the duct aspect ratio (H/W). The functions a^*, b^*, and f^* are shown in Table 1.

From Eqs. (18) and (21), Eq. (22) can be rewritten as follows.

$$\tau_w = K\left[\dot{\gamma}_a\left(\frac{2}{3}\right)\left(\frac{b^*}{f^*}+\frac{a^*}{f^*}\frac{1}{n}\right)\right]^n \tag{23}$$

Therefore, the wall shear rate of a rectangular duct is

$$\dot{\gamma}_w = \dot{\gamma}_a\left(\frac{2}{3}\right)\left(\frac{b^*}{f^*}+\frac{a^*}{f^*}\frac{1}{n}\right) \tag{24}$$

Table 1
Geometric constants a^*, b^*, and f^* for rectangular ducts [11]

H/W	a^*	b^*	f^*
0.00	0.5000	1.0000	1.0000
0.05	0.4535	0.9513	0.9365
0.10	0.4132	0.9098	0.8820
0.15	0.3781	0.8745	0.8351
0.20	0.3475	0.8444	0.7946
0.25	0.3212	0.8183	0.7597
0.30	0.2991	0.7954	0.7297
0.35	0.2809	0.7750	0.7040
0.40	0.2659	0.7571	0.6820
0.45	0.2538	0.7414	0.6634
0.50	0.2439	0.7278	0.6478
0.55	0.2360	0.7163	0.6348
0.60	0.2297	0.7065	0.6242
0.65	0.2248	0.6985	0.6155
0.70	0.2208	0.6921	0.6085
0.75	0.2178	0.6870	0.6032
0.80	0.2155	0.6831	0.5991
0.85	0.2139	0.6803	0.5961
0.90	0.2129	0.6785	0.5942
0.95	0.2123	0.6774	0.5931
1.00	0.2121	0.6771	0.5928

The functions a^*, b^*, and f^* have the following relation

$$f^* = \frac{2}{3}(a^* + b^*) \tag{25}$$

When the power-law index approaches unity (for Newtonian fluids), Eq. (24) becomes Eq. (21). When H/W approaches zero (for a slit die), Eq. (24) becomes Eq. (17).

The relation between $\dot{\gamma}_a$ and $\dot{\gamma}_w$ for non-Newtonian fluids in a rectangular duct can now be determined.

The τ_w versus $\dot{\gamma}_w$ relation from ΔP and Q data in a rectangular duct can be obtained in a simple and straightforward fashion as follows:

① Calculate τ_w and $\dot{\gamma}_a$ from ΔP and Q by the Eqs. (18) and (21).
② Plot log τ_w versus log $\dot{\gamma}_a$.
③ Obtain n from the slope of the curve.
④ Calculate $\dot{\gamma}_w$ by Eq. (24).

3. Experimental

3.1. Materials

Polystyrene (PS) and polyethylene (PE) were used in this study. PS was donated from Cheil Industries Inc. (trade name: HF2660). PE was obtained from Dow Chemicals (trade name: Engage EG8150).

3.2. Apparatus and method

Extrusion experiments with four rectangular dies having different H/W ratios as well as a capillary die were carried out using a piston-driven homemade capillary rheometer. The rheometer consists of a heated metal barrel, a piston, and a die. The barrel containing a die is attached to a universal mechanical tester (Model Hounsfield H25KS) so that the force exerted on the piston can be measured by a load cell. From the piston speed and the force, the flow rate Q and the extrusion pressure ΔP are calculated. The apparatus was described in detail by Lee et al. [12]. Four rectangular dies of various H/W and a capillary die were used. All dies were made of stainless steel. The length of all rectangular dies is 30 mm. Height-to-width ratios (H/W) are 3/3, 3/6, 2/8, and 1/10 in mm/mm, respectively. The capillary die has 1.56 mm diameter and 15 mm length. The pressure drop across the die length and the flow rate were determined by extrusion experiments and flow curves (τ_w versus $\dot{\gamma}_w$) were constructed according to the procedure described in the previous section.

4. Results and discussion

Figs. 2 and 3 are the flow curves (τ_w versus $\dot{\gamma}_w$) and shear viscosities versus $\dot{\gamma}_w$ plots by the procedure suggested in this study for the four rectangular dies and the capillary die. Shear stresses and viscosities from the five dies are in good agreement. Shear stress and shear viscosity of PS deviate slightly from the master curve at higher shear rate. The length-to-height (or length-to-diameter in the capillary die) ratio of the dies is different. Total pressure drop across the die comprises the pressure drop from the entrance region, the die land (fully developed region), and die exit. As the length-to-height ratio increases, the portion of die end effect (the sum of the entrance and exit region) decreases. Because the length-to-height ratio for the die of $H/W = 1/10$ is largest, the fact that the stress is lowest at the higher shear rate can be attributed to the end effect. However, the flow curves from all dies used in this study are characterized by a single curve at the lower shear rate ($<10\ \mathrm{s}^{-1}$). It is thus verified that the procedure proposed in this paper is reliable for constructing the flow curves and

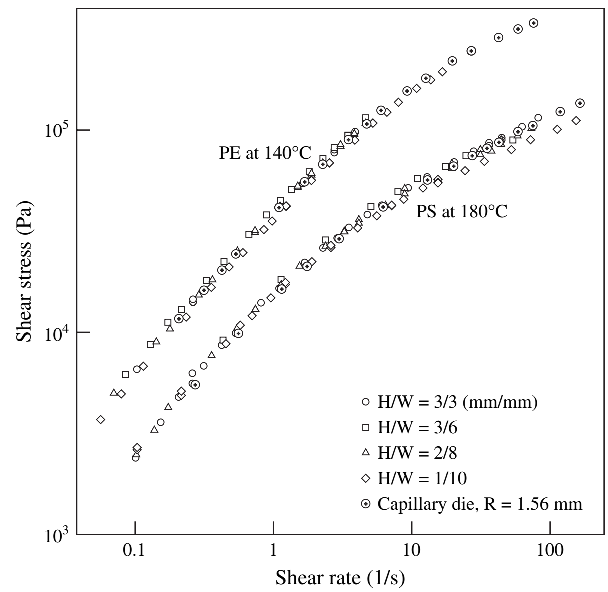

Fig. 2. Shear stress versus shear rate plot constructed by the procedure suggested in this study.

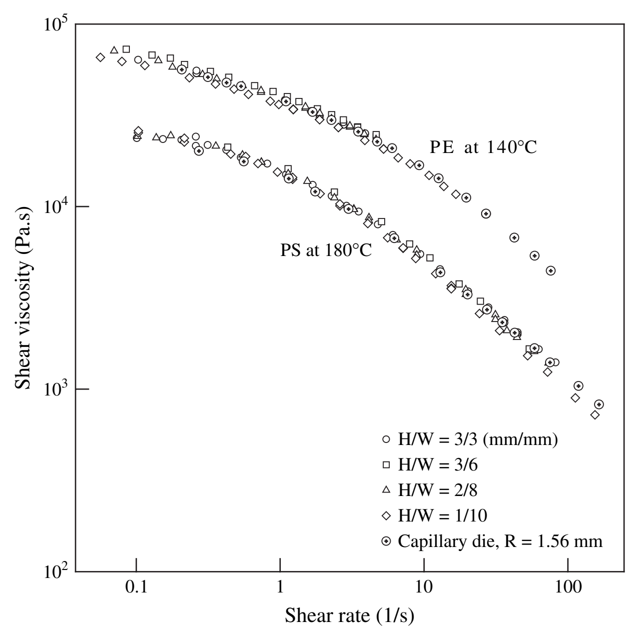

Fig. 3. Shear viscosity versus shear rate plot constructed by the procedure suggested in this study.

obtaining the viscometric properties of non-Newtonian fluids from a rectangular die having H/W near unity.

In order to assess the applicability of Eq. (17) to the rectangular die, the flow curves are constructed by Eqs. (13) and (17) from the ΔP and Q data of the rectangular dies and are shown in Fig. 4. As a reference, the flow curve for the capillary die is also included. As expected, the flow curves for the dies having a high H/W value deviate considerably from the flow curve for the capillary die. The flow curve for the die having $H/W = 1/10$ is in good agreement with that of the capillary die. We estimate the relative error in constructing the flow curve for the rectangular die under the assumption that the rectangular die is an infinitely wide slit (Fig. 5), as follows.

$$\text{Error} = \sqrt{\left(\frac{\tau_R - \tau_S}{\tau_R}\right)^2 + \left(\frac{\dot{\gamma}_R - \dot{\gamma}_S}{\dot{\gamma}_R}\right)^2} \quad (26)$$

where τ_R is the stress calculated by Eq. (18) for the rectangular die, τ_S is given by Eq. (13) for the slit die, $\dot{\gamma}_R$ is the shear rate calculated by Eq. (24) for the rectangular die, and $\dot{\gamma}_S$ is given by Eq. (17) for the slit die. According to these calculations, the rectangular dies of $W/H = 10$ and $W/H = 20$ produce approximately 10% and 5% error, respectively, as shown in Fig. 5. Based on these calculations, it is inferred that a rectangular die with a width greater than $W/H = 20$ would be suitable as a slit die. For a rectangular die of $W/H < 10$, derivation of the equation that transforms the Q and ΔP from a rectangular channel to true rheological data is expected to be valuable.

Several studies use the following relation (i.e., the equivalent radii) to obtain the shear rate of a rectangular die [2,13].

$$R_{eq} = \sqrt{\frac{HW}{\pi}} \quad (27a)$$

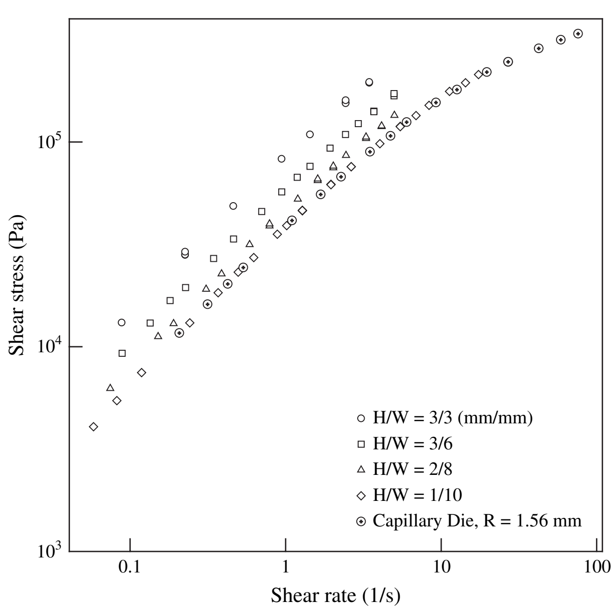

Fig. 4. Shear stress versus shear rate plot constructed with the assumption of an infinitely wide slit.

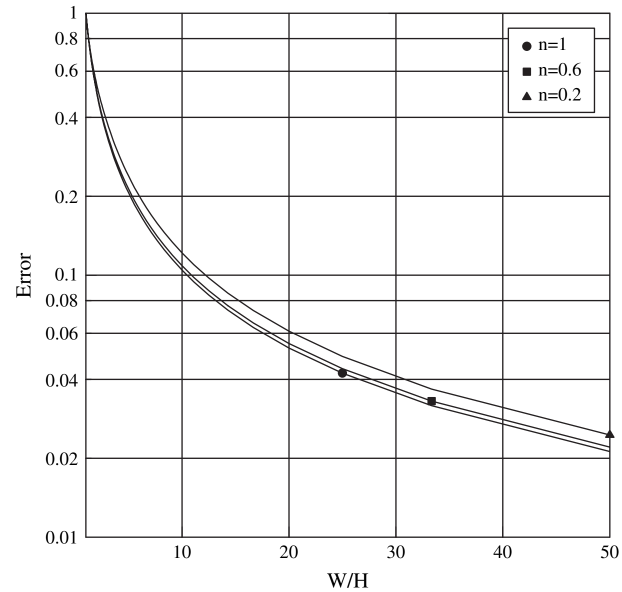

Fig. 5. Error analysis of the assumption of an infinitely wide slit.

$$\dot{\gamma}_a = \left(\frac{4Q}{\pi R_{eq}^3}\right) \quad (27b)$$

The shear stress for this approach corresponds with that of the rectangular channel (Eq. (18)). The above relation simply approximates the rectangular cross-section as a circle, and uses the shear rate equation for the circular channel. We also present the flow curves thus obtained in Fig. 6. The flow curve for the square die is in good agreement with that of the capillary die. The other rectangular dies produce large errors. Therefore, the above relation is only valid for dies having a square cross-section (i.e., $H = W$).

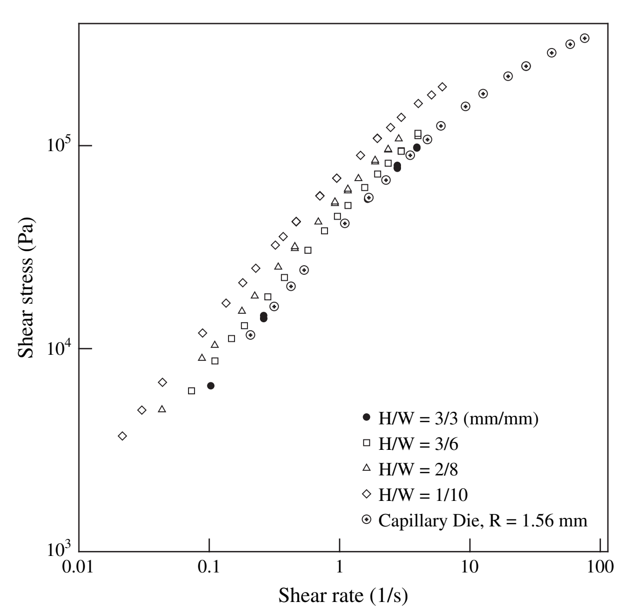

Fig. 6. Shear stress versus shear rate plot constructed with the assumption of equivalent radii.

5. Conclusions

The reliability of the suggested method in terms of calculating the shear rate, shear stress, and shear viscosity from ΔP and Q in a rectangular channel has been verified.

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